

# A DISCONTINUOUS PETROV-GALERKIN METHOD FOR TIME-FRACTIONAL DIFFUSION EQUATIONS \*

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**Abstract.** We propose and analyze a time-stepping discontinuous Petrov-Galerkin method combined with the continuous conforming finite element method in space for the numerical solution of time-fractional subdiffusion problems. We prove the existence, uniqueness and stability of approximate solutions, and derive error estimates. To achieve high order convergence rates from the time discretizations, the time mesh is graded appropriately near  $t = 0$  to compensate the singular (temporal) behaviour of the exact solution near  $t = 0$  caused by the weakly singular kernel, but the spatial mesh is quasiuniform. In the  $L_\infty((0, T); L_2(\Omega))$ -norm ( $(0, T)$  is the time domain and  $\Omega$  is the spatial domain), for sufficiently graded time meshes, a global convergence of order  $k^{m+\alpha/2} + h^{r+1}$  is shown, where  $0 < \alpha < 1$  is the fractional exponent,  $k$  is the maximum time step,  $h$  is the maximum diameter of the spatial finite elements, and  $m$  and  $r$  are the degrees of approximate solutions in time and spatial variables, respectively. Numerical experiments indicate that our theoretical error bound is pessimistic. We observe that the error is of order  $k^{m+1} + h^{r+1}$ , that is, optimal in both variables.

**Key words.** Fractional diffusion, discontinuous Petrov-Galerkin method, variable time steps, stability and error analysis

**1. Introduction.** In this paper, we propose and analyze the time-stepping discontinuous Petrov-Galerkin (DPG) method combined with the standard continuous finite element (DPG-FE) method in space for solving numerically the time-fractional diffusion model:

$${}^cD^{1-\alpha}u(x, t) - \Delta u(x, t) = f(x, t) \quad \text{for } (x, t) \in \Omega \times (0, T] \quad \text{with } u(x, 0) = u_0(x), \quad (1.1)$$

subject to homogeneous Dirichlet spatial boundary conditions. Here,  $\Omega \subset \mathbb{R}^d$  (with  $d = 1, 2, 3$ ) is a convex polyhedral domain with boundary  $\partial\Omega$ ,  $f$  and  $u_0$  are given functions assumed to be sufficiently regular such that the solution  $u$  of (1.1) is in the space  $W^{1,1}((0, T); H^2(\Omega))$ ; see the regularity analysis in [18] (further regularity assumptions will be imposed later), and  $T > 0$  is a fixed value. Here,  ${}^cD^{1-\alpha}$  denotes the time fractional Caputo derivative of order  $\alpha$  of the function  $u$  defined by

$${}^cD^{1-\alpha}u(x, t) := (\mathcal{I}^\alpha u')(x, t) \quad \text{with } 0 < \alpha < 1, \quad (1.2)$$

where  $u' := \frac{\partial}{\partial t}u$  and  $\mathcal{I}^\alpha$  is the Riemann–Liouville time fractional integral operator;

$$\mathcal{I}^\alpha v(t) := \int_0^t \omega_\alpha(t-s)v(s)ds \quad \text{with } \omega_\alpha(t) := \frac{t^{\alpha-1}}{\Gamma(\alpha)}. \quad (1.3)$$

Problems of the form (1.1) arise in a variety of physical, biological and chemical applications [17, 27, 30, 32]. Problem (1.1) describes slow or anomalous sub-diffusion and occurs, for example, in models of fractured or porous media, where the particle flux depends on the entire history of the density gradient  $\nabla u$ .

**1.1. Motivation and outline of the paper.** The nonlocal nature of the fractional derivative operator  ${}^cD^{1-\alpha}$  means that on each time subinterval, one must efficiently evaluate a sum of integrals over all previous time subintervals. Thus, reducing

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the number of time steps as much as possible and maintaining high accuracy of the discrete solutions are important. So, the most obvious thing is to propose efficient high order methods for the model problem (1.1). In this work we investigate (for the first time to the best of our knowledge) a high order accurate (unconditionally stable) time-stepping numerical method for problem (1.1). However, due to the typical singular behaviour of  $u$  near  $t \rightarrow 0$  [18, 19], high order methods can fail to achieve fast convergence. To this end, we propose to deal with the accuracy issue (in time) by developing a high order DPG method that allows for the singular behaviour of  $u$ , by employing non-uniform time steps. An important feature of the DPG method is that it allows for locally varying time-steps and approximation orders which are beneficial to handle problems with low regularity. The DPG method was introduced initially for solving first order ODEs in [9]. Later on, DPG methods were investigated by several authors for solving various problems. For instance, for advection-diffusion and elliptic problems, see [1, 2], for transport equations see [7], and refer to [14, 21] for Volterra integro-differential equations with smooth memory. Here, we extend the original DPG method in [9] to discretize in time the fractional diffusion problem (1.1). For the sake of completeness, we combine the time-stepping DPG with the continuous finite elements (FEs) in space, which will then define a fully discrete computable scheme. Existence, uniqueness and stability of our numerical scheme will be provided. For the error analysis, we show convergence rates of order  $O(k^{m+\alpha/2} + h^{r+1})$  in the  $L_\infty((0, T); L_2(\Omega))$ -norm, where  $k$  is the maximum time step,  $h$  is the maximum diameter of the spatial finite elements, and  $m$  and  $r$  are the degrees of approximate solutions in time and spatial variables, respectively. The main difficulty in our stability and error analysis is due to the trouble from the time discretization. In this direction, we make full use of several important properties of the operator  ${}^cD^{1-\alpha}$ ; see Lemma 3.1. In contrast, for  $m = 1$ , the considered time stepping DPG scheme amounts to a generalized post-processed Crank-Nicolson scheme. To validate the achieved theoretical results, a series of numerical results will be given at the end of the paper. Since in the present work our emphasis is on convergence properties rather than algorithmic implementation, in our numerical experiments we do not use the fast algorithm. A direct implementation of the considered method requires  $\mathcal{O}(mN^2\mathbf{M})$  operations and requires  $\mathcal{O}(mN\mathbf{M})$  storage, owing to the presence of the memory term, where  $N$  is the number of time mesh elements and  $\mathbf{M}$  is the spatial degrees of freedom. Proposing a fast algorithm for evaluating the discrete solution is beyond the scope of the present paper. This will be a topic for future research.

The outline of the paper is as follows. Section 2 introduces a fully discrete DPG-FE scheme. In Section 3, using appropriately the positivity, coercivity, and continuity properties of the operator  ${}^cD^{1-\alpha}$ , we prove the existence, uniqueness, and stability of the discrete solution. The error and convergence analysis are given in Section 4. We derive error estimates, which are completely explicit in the local step sizes, the local polynomial degrees, and the local regularity of the analytical solution. Using suitable refined time-steps (towards  $t = 0$ ), in the  $L_\infty((0, T); L_2(\Omega))$ -norm, convergence of order  $O(k^{m+\alpha/2} + h^{r+1})$  will be achieved. Section 5 is devoted to present a series of numerical tests which indicate the validity of our theoretical convergence properties and also illustrate that our error bounds are pessimistic. For a strongly graded time mesh, we observe that the error from the time discretization is  $O(k^{m+1})$  (optimal), which is better than our theoretical estimate by a factor  $k^{1-\alpha/2}$ .

**1.2. Literature review.** Several authors have proposed a variety of low-order numerical methods for the model problem (1.1). For one dimensional cases, [38]

constructed a box-type scheme based on combining order reduction approach and  $L_1$  discretization was considered. The authors proved global convergence rates of order  $O(k^{2-\alpha} + h^2)$ , assuming that the solution  $u$  of (1.1) is sufficiently regular. An implicit finite difference scheme in time and Legendre spectral methods in space were studied in [15]. Stability and convergence of order  $O(k^{1+\alpha} + r^{-\ell})$  of the method were established, where  $r$  is the degree of the approximate solution in space and  $\ell$  is related to the order of regularity of the solution  $u$  of (1.1), which is typically low. An extension of this work was considered in [13] where a time-space spectral method has been proposed and analyzed. For an explicit difference (first order in time-second order in space) method, we refer the reader to [28]. The stability analysis was carried out by means of a kind of fractional von Neumann method. The authors provided a partial convergence analysis (truncation error of order  $O(k + h^2)$ ) assuming that  $u$  is sufficiently regular. An implicit Crank–Nicolson had been considered in [31] and the stability of the proposed scheme was shown. Some numerical experiments were presented to illustrate the convergence of the approximate solutions. Very recently, two finite difference/element approaches were developed in [36], in which the time direction was approximated by the fractional linear multistep method and the space direction was approximated by the standard FEM of degree  $r$ . Assuming the solution of (1.1) is sufficiently smooth, convergence rates of order  $O(k^{1+\alpha} + h^{r+1})$  were proved.

For two (or three) dimensional cases, a standard central difference approximation was used for the spatial discretization, and, for the time stepping, two alternating direction implicit (ADI) schemes based on the  $L_1$  approximation and backward Euler method were investigated in [37]. Assuming that  $u$  is smooth, the authors proved convergence of order  $O(k^{\min\{2\alpha, 2-\alpha\}} + h^2)$  and  $O(k^{\min\{1+\alpha, 2-\alpha\}} + h^2)$ , respectively. A compact finite difference method with operator-splitting techniques was considered in [6]. The Caputo derivative was evaluated by the  $L_1$  approximation, and the second order spatial derivatives were approximated by the fourth order compact (implicit) finite differences. The unconditional stability was analyzed, and by using the energy method, errors of order  $O(k^{\min\{1+\alpha, 2-\alpha\}} + h^4)$  were achieved assuming that  $u$  is smooth. In [10], for  $f = 0$  in problem (1.1) (that is, homogeneous case), the authors studied two spatial semidiscrete piecewise linear approximation schemes: Galerkin FEM and lumped mass Galerkin FEM. Optimal error estimates were established including the case of non-smooth initial data. In [11], the same authors developed two simple fully discrete schemes based on Galerkin FEMs in space and implicit backward differences for the time discretizations. Optimal error estimates with respect to the regularity of the initial data were established.

In contrast, for the numerical solutions of the alternative representation of the fractional subdiffusion problem (1.1):

$$u'(x, t) - {}^R D^{1-\alpha} \Delta u(x, t) = f(x, t) \quad \text{for } (x, t) \in \Omega \times (0, T], \quad (1.4)$$

where  ${}^R D^{1-\alpha} u(x, t) := \frac{\partial}{\partial t} (\mathcal{I}^\alpha u)(x, t)$  (Riemann–Liouville fractional time derivative of  $u$ ), we refer the reader to [3, 4, 5, 12, 16, 20, 22, 23, 24, 25, 34, 39]. Practically, the two representations are different ways of writing the same equation as they are equivalent under reasonable assumptions on the initial data, see [35]. However, the numerical methods obtained for each representation are formally different.

**2. Numerical scheme.** To describe our fully discrete DPG-FE method, we introduce a (possibly nonuniform) time partition of the interval  $[0, T]$  given by the points:  $0 = t_0 < t_1 < \dots < t_N = T$ . We set  $I_n = (t_{n-1}, t_n)$  and  $k_n = t_n - t_{n-1}$  for  $1 \leq n \leq N$ . Let  $S_h \subseteq H_0^1(\Omega) := \{v \in H^1(\Omega) : v = 0 \text{ on } \partial\Omega\}$  denotes the space

of continuous, piecewise polynomials of degree  $\leq r$  ( $r \geq 1$ ) with respect to a quasi-uniform partition of  $\Omega$  into conforming triangular finite elements, with maximum diameter  $h$ . Hence, the Ritz projection operator  $R_h : H_0^1(\Omega) \rightarrow S_h$  defined by

$$\langle \nabla(R_h v - v), \nabla \chi \rangle = 0 \quad \text{for all } \chi \in S_h, \quad (2.1)$$

has the approximation property: for  $v \in H^{s+1}(\Omega) \cap H_0^1(\Omega)$ ,

$$\|R_h v - v\| \leq C h^{\min\{s, r\}+1} \|v\|_{s+1}^2 \quad \text{for } r \geq 1 \text{ and } s \geq 0, \quad (2.2)$$

where, by  $\langle \cdot, \cdot \rangle$  and  $\|\cdot\|$ , we denote the  $L_2$ -inner product and the associated norm over the spatial domain  $\Omega$ . By  $\|\cdot\|_r$  we denote the standard  $H^r(\Omega)$ -norm for  $r \geq 1$ .

Next, we introduce the following spaces: for a fixed  $m \geq 1$ ,

$$\begin{aligned} \mathcal{W}(S_h) &= \{v \in \mathcal{C}([0, T]; S_h) : v|_{I_n} \in P_m(S_h) \text{ for } 1 \leq n \leq N\} \\ \mathcal{T}(S_h) &= \{v \in L_2((0, T); S_h) : v|_{I_n} \in P_{m-1}(S_h) \text{ for } 1 \leq n \leq N\} \end{aligned} \quad (2.3)$$

where  $P_m(S_h)$  denotes the space of polynomials of degree  $\leq m$  in the time variable  $t$ , with coefficients in  $S_h$ . So, for a given function  $v \in \mathcal{W}(S_h)$  then  $v' \in \mathcal{T}(S_h)$ . Here  $v'$  is a piecewise polynomial obtained by differentiating  $v$  with respect to  $t$  on each subinterval  $I_n$  for  $1 \leq n \leq N$ .

Now, we are ready to define our DPG-FE numerical scheme for problem (1.1) as follows: Find  $U_h \in \mathcal{W}(S_h)$  such that,  $U_h(0) = R_h u_0$ , and

$$\int_0^T (\langle {}^c D^{1-\alpha} U_h, X \rangle + \langle \nabla U_h, \nabla X \rangle) dt = \int_0^T \langle f, X \rangle dt \quad \forall X \in \mathcal{T}(S_h). \quad (2.4)$$

In the next section, we will show the well-posedness of our scheme.

**3. Well-posedness of the DPG-FE scheme.** In this section, we show the well-posedness of the discrete DPG-FE solutions. To be able to do this, we need to carefully use several crucial properties of the operator  ${}^c D^{1-\alpha}$ . These properties will be stated in the next lemma, we refer the reader to [26, Lemma 3.1] for the proof.

LEMMA 3.1. *For  $1 \leq j \leq n$ , let  $v|_{I_j}, w|_{I_j} \in H^1(I_j, L_2(\Omega)) \cap \mathcal{C}(\bar{I}_j, L_2(\Omega))$ . There holds:*

- (i) *If  $\max_{j=0}^n \|v(t_j)\| + \int_0^{t_n} \langle v', {}^c D^{1-\alpha} v \rangle dt = 0$ , then  $v \equiv 0$  on  $[0, t_n]$ .*
- (ii) *The coercivity property:*

$$\int_0^{t_n} \langle v', {}^c D^{1-\alpha} v \rangle dt \geq c_\alpha \int_0^{t_n} \|{}^c D^{1-\frac{\alpha}{2}} v\|^2 dt \quad \text{with } c_\alpha = \cos(\alpha\pi/2).$$

- (iii) *The continuity property: for any  $\epsilon > 0$ ,*

$$\left| \int_0^{t_n} \langle v', {}^c D^{1-\alpha} w \rangle dt \right| \leq \int_0^{t_n} \left( \frac{\epsilon}{2 c_\alpha^2} \langle v', {}^c D^{1-\alpha} v \rangle + \frac{1}{2\epsilon} \langle w', {}^c D^{1-\alpha} w \rangle \right) dt$$

*where assuming that  $\int_0^{t_n} \langle v', {}^c D^{1-\alpha} v \rangle dt$  and  $\int_0^{t_n} \langle w', {}^c D^{1-\alpha} w \rangle dt$  to be absolutely bounded should be sufficient for this property.*

Next, we prove the existence and uniqueness of the DPG-FE solution.

THEOREM 3.2. *Assume that  $f \in L_2((0, T); L_2(\Omega))$  and  $u_0 \in H_0^1(\Omega)$ . Then, the discrete solution  $U_h$  of (2.4) exists and is unique.*

*Proof.* Because of the finite dimensionality of problem (2.4) on each sub-domain  $\Omega \times I_n$ , the existence of the approximate solution  $U_h$  follows from its uniqueness. To show the uniqueness, we take  $X \equiv 0$  outside  $\Omega \times I_n$  in (2.4), then we find that

$$\int_{I_n} \left( \langle {}^c D^{1-\alpha} U_h, X \rangle + \langle \nabla U_h, \nabla X \rangle \right) dt = \int_{I_n} \langle f, X \rangle dt \quad \text{with } U_h(0) = R_h u_0. \quad (3.1)$$

Since  $U_h$  is constructed element by element (in time), it is enough to show the uniqueness on the first sub-domain  $\Omega \times I_1$ . To this end, let  $U_{h,1}$  and  $U_{h,2}$  be two solutions of (3.1) on  $\Omega \times I_1$ . By linearity, the difference  $V_h := U_{h,1} - U_{h,2}$  on  $\Omega \times I_1$  satisfies:

$$\int_0^{t_1} \left( \langle {}^c D^{1-\alpha} V_h, X \rangle + \langle \nabla V_h, \nabla X \rangle \right) dt = 0 \quad \text{for all } X \in P_{m-1}(S_h) \quad (3.2)$$

with  $V_h(0) = 0$ . Choosing  $X = V'_h \in P_{m-1}(S_h)$  yields

$$\int_0^{t_1} \langle {}^c D^{1-\alpha} V_h, V'_h \rangle dt + \frac{1}{2} \int_0^{t_1} \frac{d}{dt} \|\nabla V_h(t)\|^2 dt = 0.$$

Integrating, then using  $V_h(0) = 0$  and the positivity  $\int_0^{t_1} \langle {}^c D^{1-\alpha} V_h, V'_h \rangle dt \geq 0$ ; see property (ii) in Lemma 3.1, we conclude that  $\|\nabla V_h(t_1)\|^2 = \|\nabla V_h(0)\|^2 = 0$  and  $\int_0^{t_1} \langle {}^c D^{1-\alpha} V_h, V'_h \rangle dt = 0$ . Therefore,  $\|V_h(t_1)\| = \|V_h(0)\| = 0$  and consequently, an application of Lemma 3.1 (i) yields  $V_h \equiv 0$  on  $\Omega \times [0, t_1]$ . This completes the proof.  $\square$

In the next theorem, the stability of the DPG-FE scheme will be shown.

**THEOREM 3.3.** *Assume that  $f \in H^1((0, T); L_2(\Omega))$  and  $u_0 \in H_0^1(\Omega)$  in problem (1.1). Then, for  $1 \leq n \leq N$ , the DPG-FE solution  $U_h$  of (2.4) satisfies:*

$$\int_0^{t_n} \langle {}^c D^{1-\alpha} U_h, U'_h \rangle dt + \|\nabla U_h(t_n)\|^2 \leq \|\nabla u_0\|^2 + \frac{1}{c_\alpha^2} \int_0^{t_n} \langle f, {}^R D^\alpha f \rangle dt,$$

where  $c_\alpha$  is the constant in Lemma 3.1.

*Proof.* For  $1 \leq n \leq N$ , we choose  $X|_{[0, t_n]} = U'_h$  and zero elsewhere in (2.4), and use the identity  $f = {}^c D^{1-\alpha}(\mathcal{I}^{1-\alpha} f)$  (since  $f \in H^1((0, T); L_2(\Omega))$ ),

$$\int_0^{t_n} \left( \langle {}^c D^{1-\alpha} U_h, U'_h \rangle + \langle \nabla U_h, \nabla U'_h \rangle \right) dt = \int_0^{t_n} \langle {}^c D^{1-\alpha}(\mathcal{I}^{1-\alpha} f), U'_h \rangle dt, \quad (3.3)$$

But, from the continuity property (iii) of Lemma 3.1,

$$\begin{aligned} 2 \left| \int_0^{t_n} \langle {}^c D^{1-\alpha}(\mathcal{I}^{1-\alpha} f), U'_h \rangle dt \right| & \leq \int_0^{t_n} \left( \frac{1}{c_\alpha^2} \langle {}^c D^{1-\alpha}(\mathcal{I}^{1-\alpha} f), (\mathcal{I}^{1-\alpha} f)' \rangle + \langle {}^c D^{1-\alpha} U_h, U'_h \rangle \right) dt \\ & = \frac{1}{c_\alpha^2} \int_0^{t_n} \langle f, {}^R D^\alpha f \rangle dt + \int_0^{t_n} \langle {}^c D^{1-\alpha} U_h, U'_h \rangle dt. \end{aligned}$$

Inserting this in (3.3) and using;  $2 \int_0^{t_n} \langle \nabla U_h, \nabla U'_h \rangle dt = \|\nabla U_h(t_n)\|^2 - \|\nabla U_h(0)\|^2$ , will complete the proof.  $\square$

**4. Error Analysis.** In this section, we carry out a priori error analysis of the DPG-FE method (2.4). The starting point is to introduce a projection operator that has been used various times in the analysis of several numerical methods.

**4.1. Projection and errors.** For  $2 \leq n \leq N$  and for  $m \geq 1$ , the (Raviart-Thomas) projection operator  $\Pi : \mathcal{C}(\bar{I}_n; H^\ell(\Omega)) \rightarrow \mathcal{C}(\bar{I}_n; P_m(H^\ell(\Omega)))$  defined by:

$$\Pi u(t_j) = u(t_j) \quad \text{for } j = n-1, n \quad \text{and} \quad \int_{I_n} \langle \Pi u - u, v \rangle dt = 0 \quad \forall v \in P_{m-2}(L_2(\Omega)).$$

Here  $\ell \geq 0$ ,  $P_m(H^\ell(\Omega))$  is the space of polynomials of degree  $\leq m$  in the time variable  $t$ , with coefficients in  $H^\ell(\Omega)$ . On  $I_1$ , due to the singular behaviour of  $u$  at  $t = 0$  in the model problem (1.1), we let  $\Pi u|_{I_1}$  be a linear polynomial in the time variable that interpolates  $u$  at the end nodes;  $t_0$  and  $t_1$ .

Notice that, since  $(\Pi u)'|_{I_1}$  is independent of  $t$  and since  $(\Pi u - u)|_{t=0, t_1} = 0$ ,

$$\int_{I_1} \langle (\Pi u - u)'(t), (\Pi u)'(t) \rangle dt = 0.$$

However, for  $n \geq 2$ , an integration by parts yields

$$\int_{I_n} \langle (\Pi u - u)'(t), (\Pi u)'(t) \rangle dt = - \int_{I_n} \langle (\Pi u - u)(t), (\Pi u)''(t) \rangle dt = 0.$$

Hence, using these facts and the Cauchy-schwarz inequality, we obtain

$$\int_{I_n} \|(\Pi u)'(t)\|^2 dt = \int_{I_n} \langle u'(t), (\Pi u)'(t) \rangle dt \leq \frac{1}{2} \int_{I_n} (\|(\Pi u)'(t)\|^2 + \|u'(t)\|^2) dt.$$

Therefore, the projection operator  $\Pi$  has the following property: for  $1 \leq n \leq N$ ,

$$\int_{I_n} \|(\Pi u)'(t)\|^2 dt \leq 2 \int_{I_n} \|u'(t)\|^2 dt \quad \text{for any } u \in H^1(I_n; L^2(\Omega)). \quad (4.1)$$

In the next theorem, we state the error estimates of the projection operator  $\Pi$ . For convenience, we introduce the notations:

$$\|\phi\|_{I_n} := \|\phi\|_{L_\infty(I_n, L_2(\Omega))} = \sup_{t \in I_n} \|\phi(t)\|.$$

**THEOREM 4.1.** *For  $u|_{I_n} \in H^{m+1}(I_n; L_2(\Omega))$  with  $2 \leq n \leq N$ , we have*

$$\|\Pi u - u\|_{I_n}^2 + k_n^2 \|(\Pi u - u)'\|_{I_n}^2 \leq C_m k_n^{2m+1} \int_{I_n} \|u^{(m+1)}(t)\|^2 dt \quad \text{for } m \geq 1.$$

*Proof.* First, for  $m = 1$ , on the subinterval  $I_n$ ,  $\Pi u$  is a linear polynomial in time that interpolates  $u$  at the end points of  $I_n$ . Thus, for  $t \in I_n$ ,

$$\Pi u(t) = u(t) + \frac{1}{k_n} \int_{I_n} \int_t^{t_n} [u'(s) - u'(q)] ds dq \quad \text{and} \quad (\Pi u)'(t) = \frac{1}{k_n} \int_{I_n} u'(s) ds. \quad (4.2)$$

Using this representation of  $\Pi u$ , we can easily derive the desired estimate.

For  $m \geq 2$ , we recall first the following error estimate properties of the projection operator  $\Pi$  (refer for example to [29, Chapter 3] for the proof):

$$\int_{I_n} \|(\Pi u - u)'(t)\|^2 dt \leq C_m k_n^{2m} \int_{I_n} \|u^{(m+1)}(t)\|^2 dt,$$

Then, by the equality:  $(\Pi u - u)(t) = \int_{t_{n-1}}^t (\Pi u - u)'(s) ds$ , the Cauchy-Schwarz inequality, and the above estimate, we have

$$\begin{aligned} \|\Pi u - u\|_{I_n}^2 &\leq \left( \int_{I_n} \|(\Pi u - u)'(s)\| ds \right)^2 \\ &\leq k_n \int_{I_n} \|(\Pi u - u)'(s)\|^2 ds \leq C_m k_n^{2m+1} \int_{I_n} \|u^{(m+1)}(t)\|^2 dt. \end{aligned}$$

To estimate  $\|(\Pi u - u)'\|_{I_n}$ , we decompose it as:

$$\|(u - \Pi u)'\|_{I_n} \leq \|(u - \tilde{\Pi} u)'\|_{I_n} + \|(\tilde{\Pi} u - \Pi u)'\|_{I_n} \quad (4.3)$$

where  $\tilde{\Pi} u|_{I_n} \in P_m(L_2(\Omega))$  will be defined such that

$$\|u - \tilde{\Pi} u\|_{I_n}^2 + k_n^2 \|(u - \tilde{\Pi} u)'\|_{I_n}^2 \leq C_m k_n^{2m+1} \int_{I_n} \|u^{(m+1)}(t)\|^2 dt.$$

For instance, one may choose  $\tilde{\Pi} u|_{I_n}$  as follows:  $\tilde{\Pi} u$  interpolates  $u$  at  $t_{n-1}$  and  $t_n$ ,

$$(\tilde{\Pi} u)'(t_n) = u'(t_n) \quad \text{and} \quad \tilde{\Pi} u(\xi_{n,\ell}) = u(\xi_{n,\ell}), \quad \ell = 1, \dots, m-2,$$

where  $\xi_{n,\ell} = t_{n-1} + k_n \xi_\ell$  and  $0 < \xi_1 < \xi_2 < \dots < \xi_{m-2} < 1$  are the  $(m-2)$ -point Gauss-Legendre quadrature on the interval  $(0, 1)$ .

Since  $\|(\tilde{\Pi} u - \Pi u)'\|_{I_n} \leq C_m k_n^{-1} (\|\tilde{\Pi} u - u\|_{I_n} + \|u - \Pi u\|_{I_n})$  (by the inverse and triangle inequalities), from (4.3), we have

$$\begin{aligned} \|(\Pi u - u)'\|_{I_n}^2 &\leq 2\|(u - \tilde{\Pi} u)'\|_{I_n}^2 + C_m k_n^{-2} (\|\tilde{\Pi} u - u\|_{I_n}^2 + \|u - \Pi u\|_{I_n}^2) \\ &\leq C_m k_n^{2m-1} \int_{I_n} \|u^{(m+1)}(t)\|^2 dt \end{aligned}$$

and therefore, the proof is completed now.  $\square$

**4.2. Error decomposition and an interesting bound.** To estimate the error  $U_h - u$ , we decompose it into three terms (using the operators  $R_h$  and  $\Pi$ ) as follows:

$$U_h - u = \zeta + \Pi \xi + \eta := (U_h - \Pi R_h u) + \Pi(R_h u - u) + (\Pi u - u). \quad (4.4)$$

Since the Ritz projection estimate in (2.2) and the first estimate in Theorem 4.1 can be used to bound  $\Pi \xi$  and  $\eta$ , the main task reduces to bound  $\zeta$ . To do so, we derive next an interesting upper bound of  $\zeta$  that depends on  $\eta$  and  $\xi$  where we assume that  $u \in W^{1,1}((0, T); H^2(\Omega))$ . To satisfy this property we let  $u_0 \in H_0^1(\Omega) \cap H^{2+\epsilon_1}(\Omega)$  and  $\int_0^t s^j \|\frac{\partial^j}{\partial s^j} {}^R D^\alpha f(s)\|_2 ds \leq t^{\epsilon_2}$  for some  $\epsilon_1, \epsilon_2 > 0$  with  $j = 0, 1, 2$ . One way to see this is to rewrite the model problem (1.1) as:  $u' - {}^R D^\alpha \Delta u = {}^R D^\alpha f$  and then we refer to [18, Theorems 4.4 and 5.7].

**THEOREM 4.2.** *Assume that  $u \in W^{1,1}((0, T); H^2(\Omega))$ . Then, for  $1 \leq n \leq N$ , we have*

$$\begin{aligned} &\int_0^{t_n} \langle {}^c D^{1-\alpha} \zeta, \zeta' \rangle dt + \|\nabla \zeta(t_n)\|^2 \\ &\leq \frac{4}{c_\alpha^2} \left( \int_0^{t_n} (\langle {}^c D^{1-\alpha} \eta, \eta' \rangle + \langle \Delta \eta, {}^c D^\alpha \Delta \eta \rangle + \omega_{\alpha+1}^2(t_n) \|\xi'\|^2) dt \right). \quad (4.5) \end{aligned}$$

*Proof.* The DPG-FE scheme (2.4) and the decomposition in (4.4) imply

$$\begin{aligned} \int_0^T (\langle {}^cD^{1-\alpha}\zeta, X \rangle + \langle \nabla\zeta, \nabla X \rangle) dt \\ = - \int_0^T (\langle {}^cD^{1-\alpha}(\Pi\xi + \eta), X \rangle + \langle \nabla(\Pi\xi + \eta), \nabla X \rangle) dt. \end{aligned}$$

But,  $\Pi$  commutes with  $R_h$  ( $\Pi R_h = R_h \Pi$ ) and so, from the definition of Ritz projector, we have  $\langle \nabla \Pi \xi, \nabla X \rangle = \langle \nabla(R_h(\Pi u) - \Pi u), \nabla X \rangle = 0$ . Hence,

$$\int_0^T (\langle {}^cD^{1-\alpha}\zeta, X \rangle + \langle \nabla\zeta, \nabla X \rangle) dt = - \int_0^T \langle {}^cD^{1-\alpha}(\Pi\xi + \eta) - \Delta\eta, X \rangle dt.$$

Now, choosing  $X|_{(0,t_n)} = \zeta'$  and zero elsewhere, then using  $2 \int_0^{t_n} \langle \nabla\zeta, \nabla X \rangle dt = \|\nabla\zeta(t_n)\|^2 - \|\nabla\zeta(0)\|^2 = \|\nabla\zeta(t_n)\|^2$ , and the identity  $\Delta\eta(t) = {}^cD^{1-\alpha}(\mathcal{I}^{1-\alpha}\Delta\eta)(t)$  for  $t \in [0, t_n]$  (follows because  $\Delta\eta \in W^{1,1}((0, t_n), L_2(\Omega))$ ), we obtain

$$2 \int_0^{t_n} \langle {}^cD^{1-\alpha}\zeta, \zeta' \rangle dt + \|\nabla\zeta(t_n)\|^2 = -2 \int_0^{t_n} \langle {}^cD^{1-\alpha}(\Pi\xi + \eta - \mathcal{I}^{1-\alpha}\Delta\eta), \zeta' \rangle dt. \quad (4.6)$$

Now, using the continuity property, Lemma 3.1 (iii) (with  $\epsilon = 4$ ), we notice that

$$\begin{aligned} \left| \int_0^{t_n} \langle {}^cD^{1-\alpha}\eta, \zeta' \rangle dt \right| &\leq \frac{2}{c_\alpha^2} \int_0^{t_n} \langle {}^cD^{1-\alpha}\eta, \eta' \rangle dt + \frac{1}{8} \int_0^{t_n} \langle {}^cD^{1-\alpha}\zeta, \zeta' \rangle dt, \\ \left| \int_0^{t_n} \langle {}^cD^{1-\alpha}\Pi\xi, \zeta' \rangle dt \right| &\leq \frac{2}{c_\alpha^2} \int_0^{t_n} \langle {}^cD^{1-\alpha}\Pi\xi, (\Pi\xi)' \rangle dt + \frac{1}{8} \int_0^{t_n} \langle {}^cD^{1-\alpha}\zeta, \zeta' \rangle dt, \end{aligned}$$

and (with  $\epsilon = 2$ ),

$$\begin{aligned} \left| \int_0^{t_n} \langle {}^cD^{1-\alpha}(\mathcal{I}^{1-\alpha}\Delta\eta), \zeta' \rangle dt \right| \\ \leq \frac{1}{c_\alpha^2} \int_0^{t_n} \langle {}^cD^{1-\alpha}(\mathcal{I}^{1-\alpha}\Delta\eta), (\mathcal{I}^{1-\alpha}\Delta\eta)' \rangle dt + \frac{1}{4} \int_0^{t_n} \langle {}^cD^{1-\alpha}\zeta, \zeta' \rangle dt \\ = \frac{1}{c_\alpha^2} \int_0^{t_n} \langle \Delta\eta, {}^cD^\alpha\Delta\eta \rangle dt + \frac{1}{4} \int_0^{t_n} \langle {}^cD^{1-\alpha}\zeta, \zeta' \rangle dt, \end{aligned}$$

where in the second inequality, we used the identity:

$$(\mathcal{I}^{1-\alpha}\Delta\eta)'(t) = \omega_{1-\alpha}(t) \Delta\eta(0) + {}^cD^\alpha\Delta\eta(t) = {}^cD^\alpha\Delta\eta(t).$$

Inserting the above inequalities in (4.6) and rearranging the terms yield

$$\begin{aligned} \int_0^{t_n} \langle {}^cD^{1-\alpha}\zeta, \zeta' \rangle dt + \|\nabla\zeta(t_n)\|^2 \\ \leq \frac{4}{c_\alpha^2} \int_0^{t_n} (\langle {}^cD^{1-\alpha}\eta, \eta' \rangle + \langle \Delta\eta, {}^cD^\alpha\Delta\eta \rangle + \langle {}^cD^{1-\alpha}\Pi\xi, (\Pi\xi)' \rangle) dt. \quad (4.7) \end{aligned}$$

To complete the proof, we still need to estimate the third term on the right-hand side of (4.7). By the Cauchy-Schwarz inequality, the inequality:

$$\left( \int_0^{t_n} \|{}^cD^{1-\alpha}\Pi\xi\|^2 dt \right)^{1/2} \leq \omega_{\alpha+1}(t_n) \left( \int_0^{t_n} \|(\Pi\xi)'\|^2 dt \right)^{1/2},$$



and the property of the operator  $\Pi$  in (4.1) (with  $\Pi\xi$  in place of  $\Pi u$ ), we have

$$\begin{aligned} \left| \int_0^{t_n} \langle {}^c D^{1-\alpha} \Pi \xi, \Pi \xi' \rangle dt \right| &\leq \int_0^{t_n} \| {}^c D^{1-\alpha} \Pi \xi \| \| (\Pi \xi)' \| dt \\ &\leq \left( \int_0^{t_n} \| {}^c D^{1-\alpha} \Pi \xi \|^2 dt \right)^{1/2} \left( \int_0^{t_n} \| (\Pi \xi)' \|^2 dt \right)^{1/2} \\ &\leq \omega_{\alpha+1}^2(t_n) \int_0^{t_n} \| (\Pi \xi)' \|^2 dt \leq \omega_{\alpha+1}^2(t_n) \int_0^{t_n} \| \xi' \|^2 dt. \end{aligned}$$

Finally, the desired inequality is obtained after inserting this estimate in (4.7).  $\square$

**4.3. Regularity and time meshes.** As mentioned earlier, the solution  $u$  of the fractional model problem (1.1) has a singular behaviour near  $t = 0$ . Under suitable regularity assumptions on the initial data  $u_0$  and the forcing term  $f$  in problem (1.1),  $u$  satisfies: for  $t > 0$  and for  $1 \leq q \leq m + 1$ ,

$$\| u^{(q)}(t) \| \leq c_q t^{\sigma-q} \quad \text{and} \quad \| \Delta u^{(q)}(t) \| \leq d_q t^{\delta-q-1} \quad (4.8)$$

for some positive constants  $c_q$  and  $d_q$ , with  $(1 - \alpha)/2 < \sigma < 1$  and  $\delta > 1$ . The proof of (4.8) follows from the regularity analysis in [18, 19].

Because  $u$  is not sufficiently smooth near  $t = 0$ , the global error in  $U_h$  fails to be  $O(k^{m+1})$  accurate in time if we use a uniform time step  $k$ . Typically, for high order methods over uniform time meshes, one should not expect to observe global convergence rates of an order better than  $O(k^\sigma)$  in the  $L_\infty(0, T)$ -norm. Now, to capture the singular behaviour of  $u$  near  $t = 0$ , following [19, 20, 24, 25], we employ a family of non-uniform meshes that concentrate the time levels near  $t = 0$ . More precisely, we assume that for a fixed parameter  $\gamma \geq 1$ ,

$$t_n = (nk)^\gamma \quad \text{with} \quad k = \frac{T^{1/\gamma}}{N} \quad \text{for } 0 \leq n \leq N. \quad (4.9)$$

Noting that the time step sizes are nondecreasing, that is,  $k_i \leq k_j$  for  $i \leq j$ . For  $2 \leq n \leq N$  one can show that

$$\frac{\gamma}{2^{\gamma-1}} k t_n^{1-\frac{1}{\gamma}} \leq k_n \leq \gamma k t_n^{1-\frac{1}{\gamma}} \quad \text{and} \quad t_n \leq 2^\gamma t_{n-1}. \quad (4.10)$$

The aim now is to bound the first and second terms on the right-hand side of (4.5) in Theorem 4.2.

**4.4. Estimate of  $\int_0^{t_n} \langle {}^c D^{1-\alpha} \eta, \eta' \rangle dt$ .** Assume that  $u$  satisfies the first regularity assumption in (4.8). Then, there exists a positive constant  $C$  that depends on  $d_1$ ,  $\sigma$ ,  $\alpha$ ,  $\gamma$ ,  $m$  and  $T$ , such that, for  $1 \leq n \leq N$ ,

$$\int_0^{t_n} \langle {}^c D^{1-\alpha} \eta, \eta' \rangle dt \leq C k^{2m+\alpha} \quad \text{for} \quad \gamma \geq (2m + 1 + \alpha)/(\alpha + 2\sigma - 1).$$

*Proof.* We start our proof by splitting  $\int_0^{t_n} \langle {}^c D^{1-\alpha} \eta, \eta' \rangle dt$  as follows:

$$\int_0^{t_n} \langle {}^c D^{1-\alpha} \eta, \eta' \rangle dt = \int_{I_1} \langle \mathcal{A}_1^\alpha(t), \eta'(t) \rangle dt + \sum_{j=2}^n \int_{I_j} \langle \mathcal{A}_2^\alpha(t) + \mathcal{A}_{3,j}^\alpha(t), \eta'(t) \rangle dt, \quad (4.11)$$

where

$$\begin{aligned}
\mathcal{A}_1^\alpha(t) &:= \int_0^t \omega_\alpha(t-s) \eta'(s) ds \quad \text{and} \\
\mathcal{A}_2^\alpha(t) &:= \int_0^{t_1} \omega_\alpha(t-s) \eta'(s) ds \\
\mathcal{A}_{3,j}^\alpha(t) &:= \int_{t_1}^t \omega_\alpha(t-s) \eta'(s) ds \\
&= - \int_{t_1}^{t_{j-1}} \omega_{\alpha-1}(t-s) \eta(s) ds + \int_{t_{j-1}}^t \omega_\alpha(t-s) \eta'(s) ds \quad \text{for } t \in I_j \text{ with } j \geq 2.
\end{aligned}$$

For  $t \in I_1$ , from (4.2) (with  $n = 1$ ) and the first regularity property in (4.8) ( $\sigma < 1$ ), we observe

$$\begin{aligned}
\|\eta'(t)\| &\leq \frac{1}{t_1} \int_0^{t_1} \|u'(s)\| ds + \|u'(t)\| \\
&\leq \frac{C}{t_1} \int_0^{t_1} s^{\sigma-1} ds + Ct^{\sigma-1} \leq C(t_1^{\sigma-1} + t^{\sigma-1}) \leq Ct^{\sigma-1}.
\end{aligned} \tag{4.12}$$

Hence, using the Cauchy-Schwarz inequality and integrating, we have

$$\begin{aligned}
\int_{I_1} |\langle \mathcal{A}_1^\alpha(t), \eta'(t) \rangle| dt &\leq \int_{I_1} \|\eta'(t)\| \int_0^t \omega_\alpha(t-s) \|\eta'(s)\| ds dt \\
&\leq C \int_{I_1} t^{\sigma-1} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} s^{\sigma-1} ds dt \\
&= C \frac{\Gamma(\sigma)}{\Gamma(\sigma+\alpha)} \int_{I_1} t^{\sigma-1} t^{\sigma+\alpha-1} dt \leq Ct_1^{\alpha+2\sigma-1}.
\end{aligned} \tag{4.13}$$

To estimate the term involving  $\mathcal{A}_2^\alpha(t)$ , we use (4.12), and the inequality:  $(t_j - s)^\alpha - (t_{j-1} - s)^\alpha \leq k_j^\alpha$  after integrating,

$$\begin{aligned}
\int_{I_j} |\langle \mathcal{A}_2^\alpha(t), \eta'(t) \rangle| dt &\leq \int_{I_j} \|\eta'(t)\| \int_{I_1} \omega_\alpha(t-s) \|\eta'(s)\| ds dt \\
&\leq C \|\eta'\|_{I_j} \int_{I_1} \int_{I_j} (t-s)^{\alpha-1} s^{\sigma-1} dt ds \\
&\leq C \|\eta'\|_{I_j} k_j^\alpha t_1^\sigma = C k_j \|\eta'\|_{I_j} k_j^{\alpha-1} t_1^\sigma \quad \text{for } j \geq 2.
\end{aligned}$$

Hence, summing over  $j$ , and using  $k_j^{(\alpha-1)/2} \leq k_1^{(\alpha-1)/2}$  for  $j \geq 1$  (from the mesh properties), we get

$$\begin{aligned}
\sum_{j=2}^n \int_{I_j} |\langle \mathcal{A}_2^\alpha(t), \eta'(t) \rangle| dt &\leq C \sum_{j=2}^n (k_j^{(\alpha+1)/2} \|\eta'\|_{I_j}) t_1^{\sigma+(\alpha-1)/2} \\
&\leq C \left( \max_{j=2}^n (k_j^{\alpha+1} \|\eta'\|_{I_j}^2) + t_1^{\alpha+2\sigma-1} \right) k^{-1}.
\end{aligned} \tag{4.14}$$

It remains to estimate the term  $\int_{I_j} \langle \mathcal{A}_{3,j}^\alpha(t), \eta'(t) \rangle dt$ . Splitting it into two terms, changing the order of integrals and using;  $\int_{t_1}^{t_{j-1}} [\omega_\alpha(t_{j-1} - s) - \omega_\alpha(t_j - s)] ds \leq$

$\omega_{\alpha+1}(k_j)$ , we notice that

$$\begin{aligned} \int_{I_j} \|\mathcal{A}_{3,j}^\alpha(t)\| dt &\leq \int_{I_j} \left( \int_{t_1}^{t_{j-1}} \omega_{\alpha-1}(t-s) \|\eta(s)\| ds + \int_{t_{j-1}}^t \omega_\alpha(t-s) \|\eta'(s)\| ds \right) dt \\ &\leq \int_{t_1}^{t_{j-1}} [\omega_\alpha(t_{j-1}-s) - \omega_\alpha(t_j-s)] \|\eta(s)\| ds + \omega_{\alpha+2}(k_j) \|\eta'\|_{I_j} \\ &\leq \omega_{\alpha+1}(k_j) \max_{i=2}^{j-1} \|\eta\|_{I_i} + \omega_{\alpha+2}(k_j) \|\eta'\|_{I_j}. \end{aligned}$$

Therefore, summing over  $j$ , then, using the Cauchy-Schwarz inequality, the inequality  $k_j^{\alpha-1} \leq k_i^{\alpha-1}$  for  $i \leq j$ , and the identity  $\eta(t) = \int_{t_{i-1}}^t \eta'(s) ds$  for  $t \in I_i$  (because  $\eta(t_i) = 0$  for  $i = 0, 1, \dots, N$ ), we observe

$$\begin{aligned} \int_{t_1}^{t_n} |\langle \mathcal{A}_{3,j}^\alpha(t), \eta'(t) \rangle| dt &\leq C \sum_{j=2}^n \left( k_j^{\frac{\alpha+1}{2}} \|\eta'\|_{I_j} k_j^{\frac{\alpha-1}{2}} \max_{i=2}^{j-1} \|\eta\|_{I_i} + k_j^{\alpha+1} \|\eta'\|_{I_j}^2 \right) \\ &\leq C \sum_{j=2}^n \left( k_j^{\alpha-1} \max_{i=2}^{j-1} \|\eta\|_{I_i}^2 + k_j^{\alpha+1} \|\eta'\|_{I_j}^2 \right) \\ &\leq C k^{-1} \max_{j=2}^n (k_j^{\alpha+1} \|\eta'\|_{I_j}^2). \end{aligned} \quad (4.15)$$

Now, from the interpolation errors in Theorem 4.1, the first regularity assumption in (4.8), the time mesh property (4.10), and the graded exponent time mesh assumption,  $\gamma \geq (2m+1+\alpha)/(2\sigma+\alpha-1)$ , we get

$$\begin{aligned} k_j^{\alpha+1} \|\eta'\|_{I_j}^2 &\leq C k_j^{2m+\alpha} \int_{I_j} \|u^{(m+1)}(t)\|^2 dt \\ &\leq C k_j^{2m+1+\alpha} t_j^{2(\sigma-m-1)} \\ &\leq C k^{2m+1+\alpha} t_j^{2m+1+\alpha-(2m+1+\alpha)/\gamma} t_j^{2(\sigma-m-1)} \\ &= C k^{2m+1+\alpha} t_j^{2\sigma+\alpha-1-(2m+1+\alpha)/\gamma} \leq C k^{2m+1+\alpha}. \end{aligned} \quad (4.16)$$

Finally, combining (4.11)–(4.15), the above bound, and using the inequality  $t_1^{2\sigma+\alpha-1} = k^{\gamma(2\sigma+\alpha-1)} \leq k^{m+\alpha+1}$ , yield the desired estimate.  $\square$

**4.5. Estimate of  $\int_0^{t_n} \langle {}^c D^\alpha \Delta \eta, \Delta \eta \rangle dt$ .** Assume that  $u$  satisfies the second regularity assumption in (4.8). Then, for  $1 \leq n \leq N$ , we have

$$\int_0^{t_n} \langle {}^c D^\alpha \Delta \eta, \Delta \eta \rangle dt \leq C k^{2m+1} \text{ for } \gamma \geq (m+1)/(\delta-1), \quad (4.17)$$

where the constant  $C$  depends on  $d_2, \delta, \alpha, \gamma, m$  and  $T$ .

*Proof.* Following the decomposition in (4.11),

$$\int_0^{t_n} \langle {}^c D^\alpha \Delta \eta, \Delta \eta \rangle dt = \int_{I_1} \langle \Delta \mathcal{A}_1^{1-\alpha}, \Delta \eta \rangle dt + \int_{t_1}^{t_n} \langle \Delta (\mathcal{A}_2^{1-\alpha} + \mathcal{A}_{3,j}^{1-\alpha}), \Delta \eta \rangle dt. \quad (4.18)$$

To bound the first term, we use  $\Delta \mathcal{A}_1^{1-\alpha}(t) = \frac{\partial}{\partial t} \int_0^t \omega_{1-\alpha}(t-s) \Delta \eta(s) ds$  for  $t \in I_1$  (because  $\eta(0) = 0$ ) and then, integrating by parts and using the interpolation properties of the projection operator  $\Pi$ , yield

$$\int_{I_1} \langle \Delta \mathcal{A}_1^{1-\alpha}(t), \Delta \eta(t) \rangle dt = \int_{I_1} \langle \Delta \eta'(t), \mathcal{I}^{1-\alpha} \Delta \eta(t) \rangle dt = \int_{I_1} \langle \Delta \eta'(t), \mathcal{I}^{2-\alpha} \Delta \eta'(t) \rangle dt.$$

Following the steps in (4.12) and using the second regularity property in (4.8), we obtain

$$\begin{aligned} \|\Delta\eta'(t)\| &\leq \frac{1}{t_1} \int_0^{t_1} \|\Delta u'(s)\| ds + \|\Delta u'(t)\| \\ &\leq \frac{C}{t_1} \int_0^{t_1} s^{\delta-2} ds + Ct^{\delta-2} \leq C(t_1^{\delta-2} + t^{\delta-2}) \leq C \begin{cases} t^{\delta-2} & \text{for } 1 < \delta < 2 \\ t_1^{\delta-2} & \text{for } \delta \geq 2. \end{cases} \end{aligned}$$

Hence, an application of the Cauchy-Schwarz inequality followed by direct integrations, yield

$$\begin{aligned} \int_{I_1} |\langle \Delta \mathcal{A}_1^{1-\alpha}(t), \Delta\eta(t) \rangle| dt &\leq \int_{I_1} \|\Delta\eta'(t)\| \int_0^t \omega_{2-\alpha}(t-s) \|\Delta\eta'(s)\| ds dt \\ &\leq C \int_{I_1} t^{\delta-2} \int_0^t (t-s)^{1-\alpha} s^{\delta-2} ds dt \leq Ct_1^{2\delta-\alpha-1} \leq Ct_1^{2(\delta-1)} \end{aligned} \quad (4.19)$$

for  $1 < \delta < 2$ , where in the last inequality, we used  $t_1^{-\alpha} \leq Ct_1^{-1}$  since  $0 < \alpha < 1$ . A similar bound can be achieved when  $\delta \geq 2$ .

In a similar manner (see the steps used to obtain (4.14))

$$\begin{aligned} \int_{I_j} |\langle \Delta \mathcal{A}_2^{1-\alpha}(t), \Delta\eta(t) \rangle| dt &\leq \int_{I_j} \|\Delta\eta(t)\| \int_0^{t_1} \omega_{1-\alpha}(t-s) \|\Delta\eta'(s)\| ds dt \\ &\leq C \|\Delta\eta\|_{I_j} \int_{I_j} \int_0^{t_1} \omega_{1-\alpha}(t-s) s^{\delta-2} ds dt \\ &= C k_j \|\Delta\eta'\|_{I_j} \int_0^{t_1} [\omega_{2-\alpha}(t_j-s) - \omega_{2-\alpha}(t_{j-1}-s)] s^{\delta-2} ds \\ &= C k_j \omega_{2-\alpha}(k_j) \|\Delta\eta'\|_{I_j} \int_0^{t_1} s^{\delta-2} ds \leq C k_j^{2-\alpha} \|\Delta\eta'\|_{I_j} t_1^{\delta-1} \end{aligned}$$

where the identity  $\Delta\eta(t) = \int_{t_{j-1}}^t \Delta\eta'(s) ds$  is also used. Thus,

$$\sum_{j=2}^n \int_{I_n} |\langle \Delta \mathcal{A}_2^{1-\alpha}(t), \Delta\eta(t) \rangle| dt \leq C \left( \max_{j=2}^n (k_j^2 \|\Delta\eta'\|_{I_j}^2) + t_1^{2(\delta-1)} \right) \sum_{j=2}^n k_j^{1-\alpha}. \quad (4.20)$$

It remains to estimate the term  $\int_{t_1}^{t_n} |\langle \Delta \mathcal{A}_{3,j}^{1-\alpha}, \Delta\eta \rangle| dt$ . Following the steps in (4.15),

$$\begin{aligned} \sum_{j=2}^n \int_{I_j} |\langle \Delta \mathcal{A}_{3,j}^{1-\alpha}, \Delta\eta \rangle| dt &\leq C \sum_{j=2}^n \left( k_j^{1-\alpha} \|\Delta\eta\|_{I_j} \max_{i=2}^{j-1} \|\Delta\eta\|_{I_i} + k_j^{2-\alpha} \|\Delta\eta'\|_{I_j} \|\Delta\eta\|_{I_j} \right) \\ &\leq C \sum_{j=2}^n k_j^{1-\alpha} \left( k_j \|\Delta\eta'\|_{I_j} \max_{i=2}^{j-1} \|\Delta\eta\|_{I_i} + k_j^2 \|\Delta\eta'\|_{I_j}^2 \right) \\ &\leq C \sum_{j=2}^n k_j^{1-\alpha} \left( \max_{i=2}^{j-1} \|\Delta\eta\|_{I_i}^2 + k_j^2 \|\Delta\eta'\|_{I_j}^2 \right) \\ &\leq C \max_{j=2}^n (k_j^2 \|\Delta\eta'\|_{I_j}^2) \sum_{j=2}^n k_j^{1-\alpha}. \end{aligned}$$

Therefore, the desired estimate in (4.17) follows from (4.18), (4.19), (4.20), the above bound, and the following two inequalities:  $\sum_{j=2}^n k_j^{1-\alpha} \leq C k^{-\alpha}$ ,  $t_1^{\delta-1} = k^{\gamma(\delta-1)} \leq k^{m+1}$  (by the mesh assumption  $\gamma \geq (m+1)/(\delta-1)$ ) and the estimate

$$\begin{aligned} k_j \|\Delta \eta'\|_{I_j} &\leq C k_j^{m+1} \|\Delta u^{(m+1)}\|_{I_j} \leq C k_j^{m+1} t_j^{\delta-m-2} \\ &\leq C k^{m+1} t_j^{m+1-(m+1)/\gamma} t_j^{\delta-m-2} = C k^{m+1} t_j^{\delta-1-\frac{m+1}{\gamma}} \leq C k^{m+1}. \end{aligned}$$

Here, we used Theorem 4.1, the second regularity assumption in (4.8), the mesh property (4.10), and the mesh assumption  $\gamma \geq (m+1)/(\delta-1)$ .  $\square$

**4.6. The error estimates.** We are now ready to obtain our main error convergence results for the DPG-FE solution. In the next theorem, we derive suboptimal algebraic rates of convergence in time (short by order  $1 - \alpha/2$  from being optimal), and optimal convergence rates in the spacial discretization provided the solution  $u$  of (1.1) is sufficiently regular. However, our numerical results illustrate an optimal convergence rate in both time and space.

**THEOREM 4.3.** *Let  $u_0 \in H^{r+1}(\Omega)$ ,  $f \in H^1((0, T); H^2(\Omega))$ , and let the solution  $u \in W^{1,1}((0, T); H^2(\Omega))$  of problem (1.1) satisfy the regularity properties in (4.8) (with  $\sigma > (1 - \alpha)/2$  and  $\delta > 1$ ). Moreover, we assume that  $u(t_1) \in H^{r+1}(\Omega)$  and  $u' \in L_2((t_1, T); H^{r+1}(\Omega))$ . Let  $U_h \in \mathcal{W}(S_h)$  be the DPG-FE approximation defined by (2.4), and assume that the time mesh graded factor  $\gamma \geq \max\left\{\frac{m+1}{\delta-1}, \frac{2m+1+\alpha}{2\sigma+\alpha-1}\right\}$ . Then,*

$$\max_{n=1}^n \|U_h - u\|_{I_n}^2 \leq C k^{2m+\alpha} + C h^{2r+2} \left( \|u_0\|_{r+1}^2 + \|u(t_1)\|_{r+1}^2 + \int_{t_1}^{t_n} \|u'(t)\|_{r+1}^2 dt \right),$$

where  $C$  is a constant that depends only on  $d_1, d_2, \alpha, \sigma, \delta, \gamma, m$  and  $T$ .

*Proof.* From the decomposition:  $u - U_h = \zeta + \Pi\xi + \eta$  (given in (4.4)),

$$\|u - U_h\|_{I_n} \leq \|\zeta\|_{I_n} + \|\Pi\xi\|_{I_n} + \|\eta\|_{I_n} \quad \text{for } 1 \leq n \leq N. \quad (4.21)$$

**Step 1:** Estimating  $\|\zeta\|_{I_n}$ . Since  $\zeta(0) = 0$ , we notice that  $\zeta(t) = \int_0^t \zeta'(s) ds = \mathcal{I}^{1-\frac{\alpha}{2}}({}^c D^{1-\frac{\alpha}{2}} \zeta)(t)$ . So, for any  $t \in I_n$ , the use of the Cauchy-Schwarz inequality yields

$$\|\zeta(t)\|^2 \leq \left( \int_0^t \omega_{1-\frac{\alpha}{2}}(t-s) \|{}^c D^{1-\frac{\alpha}{2}} \zeta(s)\| ds \right)^2 \leq \int_0^t \omega_{1-\frac{\alpha}{2}}^2(s) ds \int_0^t \|{}^c D^{1-\frac{\alpha}{2}} \zeta(s)\|^2 ds$$

and hence, by property (ii) in lemma 3.1, Theorem 4.2, and the achieved bounds in (4.11) and (4.17), we find that

$$\|\zeta\|_{I_n}^2 \leq C k^{2m+\alpha} \quad \text{for } 1 \leq n \leq N.$$

**Step 2:** Estimating  $\|\Pi\xi\|_{I_n}$ . For  $n \geq 2$ ,  $\Pi\xi(t) = \int_{t_1}^t (\Pi\xi)'(s) ds + (\Pi\xi)(t_1)$  for  $t \in I_n$ . Since,  $(\Pi\xi)(t_1) = \xi(t_1)$ , an application of the Cauchy-Schwarz inequality gives

$$\|\Pi\xi\|_{I_n}^2 \leq \left( \int_{t_1}^{t_n} \|(\Pi\xi)'(t)\| dt + \|\xi(t_1)\| \right)^2 \leq 4 \int_{t_1}^{t_n} \|(\Pi\xi)'(t)\|^2 dt + 2\|\xi(t_1)\|^2.$$

For  $n = 1$ ,  $\Pi\xi$  is linear in the time variable  $t$ . So,  $\|\Pi\xi\|_{I_1} \leq \max\{\|(\Pi\xi)'(0)\|, \|(\Pi\xi)'(t_1)\|\}$ . Thus, by (4.1) with  $\xi$  in place of  $u$  and the Ritz projection approximation error (2.2),

$$\begin{aligned} \|\Pi\xi\|_{I_n}^2 &\leq 4 \int_{t_1}^{t_n} \|\xi'(t)\|^2 dt + 2\|\xi(t_1)\|^2 + \|\xi(0)\|^2 \\ &\leq C h^{2r+2} \left( \int_{t_1}^{t_n} \|u'(t)\|_{r+1}^2 dt + \|u(t_1)\|_{r+1}^2 + \|u_0\|_{r+1}^2 \right) \quad \text{for } 1 \leq n \leq N. \end{aligned}$$

**Step 3:** Estimating  $\|\eta\|_{I_n}$ . For  $t \in I_n$ ,  $\eta(t) = \int_{t_{n-1}}^t \eta'(s) ds$ . Thus,

$$\|\eta\|_{I_n} \leq \begin{cases} \int_0^{t_1} \|\eta'(t)\| dt \leq C t_1^\sigma \leq C t_1^{\sigma + \frac{\alpha-1}{2}} \leq C k^{m + \frac{\alpha+1}{2}}, \\ k_n \|\eta'\|_{I_n} \leq C k_n^{\frac{\alpha+1}{2}} \|\eta'\|_{I_n} \leq C k^{m + \frac{\alpha}{2}} \quad \text{for } 2 \leq n \leq N \end{cases}$$

where we used (4.12) and the mesh assumption  $\gamma \geq \frac{2m+1+\alpha}{2\sigma+\alpha-1}$  in the first estimate and (4.16) in the second one.

Therefore, the desired error estimate follows from the decomposition (4.21) and the bounds in **Step 1–Step 3**.  $\square$

**5. Numerical results.** In this section, we demonstrate the validity of the derived error results when  $\Delta u = u_{xx}$ ,  $\Omega = (0, 1)$  and  $T = 1$  in the time-fractional problem (1.1). To evaluate the errors, we introduce the finer grid

$$\mathcal{G}^q = \{t_{j-1} + nk_j/q : 1 \leq j \leq N, 0 \leq n \leq q\} \quad (5.1)$$

(recall that,  $N$  is the number of time mesh subintervals). Thus, for large values of  $q$ , the error measure  $\|v\|_q := \max_{t \in \mathcal{G}^q} \|v(t)\|$  approximate the norm  $\|v\|_{L_\infty((0,T);L_2(\Omega))}$ . To compute the spatial  $L_2$ -norm, we apply a composite Gauss quadrature rule with  $(r+1)$ -points on each interval of the finest spatial mesh where  $r$  is the degree of the approximate solution in the spatial variable.

**5.1. Example 1.** We choose  $u_0$  and  $f$  such that the exact solution is  $u(x, t) = t^{\alpha+1} \sin(\pi x)$ . It can be seen that the regularity conditions in (4.8) hold for  $\sigma = \alpha + 1$  and  $\delta = \alpha + 2$ .

First, to test the accuracy of the DPG-FE scheme (2.4) (with degree  $m$  in the time variable and  $r$  in the spatial variable) on the non-uniformly time graded meshes in (4.9) for various choices of  $\gamma \geq 1$ ,  $h$  (the spatial step-size) will be chosen such that the temporal errors are dominating. Thus, from Theorem 4.3, we expect to observe convergence of order  $O(k^{m+\alpha/2})$  for  $\gamma \geq \max\left\{\frac{m+1}{\alpha+1}, \frac{2m+1+\alpha}{3\alpha+1}\right\}$ . However, the numerical results in Table 5.1 illustrate more optimistic convergence rates. We observe a uniform global error bounded by  $Ck^{\min\{\gamma(\alpha+1), m+1\}}$  for  $\gamma \geq 1$ , which is optimal for  $\gamma \geq (m+1)/(\alpha+1)$ . So, the numerical results also demonstrated that the grading mesh parameter  $\gamma$  is relaxed. The results are also displayed graphically in Figure 5.1, where we show the errors against the number of time subintervals  $N$ , in the semi-logarithmic scale. In Figure 5.2, we demonstrate the positive influence of time graded mesh power  $\gamma$  on the error that remains valid for different values of  $0.1 \leq \alpha \leq 0.9$ . The errors achieved as a function of  $\alpha$  for different values of  $\gamma$ , but for a fixed  $N = 60$  and a fixed  $m = 2$ .

Next, we test the performance of the spatial finite elements discretization (order degree  $r$ ) of the scheme (2.4). A uniform spatial mesh that consists of  $N_x$  subintervals where each is of width  $h$  will be used. The time step-size  $k$  and the degree of the time-stepping DPG discretization are chosen such that the spatial error is dominating. Hence, from Theorem 4.3, a convergence of order  $O(h^{r+1})$  is expected. We illustrated these results in Table 5.2 for  $r = 1, 2, 3$ .

**5.2. Example 2.** (Less smooth) We choose  $u_0$  and  $f$  in problem (1.1) such  $u(x, t) = t^{1-\alpha} \sin(\pi x)$  is the exact solution. It can be seen that the regularity conditions in (4.8) hold for  $\sigma = 1 - \alpha$  and  $\delta = 2 - \alpha$ . Thus, it is less smooth than Example 1 in the time variable. As in the previous example, we demonstrate tabularly and

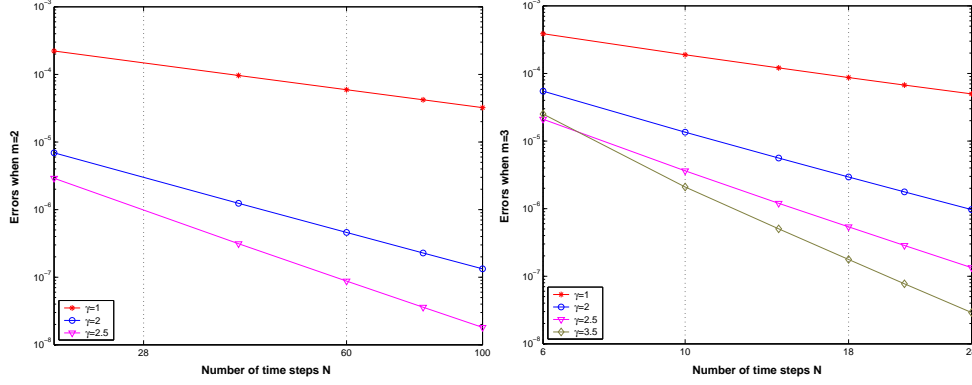


FIG. 5.1. The errors  $|||U_h - u|||_{10}$  for Example 1 plotted against  $N$  for different choices of  $\gamma$  and for  $m = 2, 3$ , with  $\alpha = 0.2$ .

$m = 1$							
$N$	$\gamma = 1$		$\gamma = 1.4$		$\gamma = 1.8$		
20	9.83e-04		2.49e-04		2.67e-04		
40	4.45e-04	1.14	8.02e-05	1.64	6.66e-05	1.99	
80	2.01e-04	1.16	2.55e-05	1.65	1.66e-05	2.00	
160	8.91e-05	1.17	8.04e-06	1.67	4.01e-06	2.05	
$m = 2$							
$N$	$\gamma = 1$		$\gamma = 2$		$\gamma = 2.5$		
20	2.22e-04		6.92e-06		2.91e-06		
40	9.64e-05	1.20	1.24e-06	2.48	3.13e-07	3.21	
60	5.93e-05	1.20	4.59e-07	2.45	8.77e-08	3.14	
80	4.21e-05	1.19	2.28e-07	2.43	3.59e-08	3.10	
100	3.22e-05	1.19	1.33e-07	2.43	1.81e-08	3.08	
$m = 3$							
$N$	$\gamma = 1$		$\gamma = 2$		$\gamma = 2.5$		$\gamma = 3.5$
10	1.89e-04	1.40	1.35e-05	2.74	3.62e-06	3.44	2.09e-06 4.87
14	1.21e-04	1.34	5.60e-06	2.62	1.20e-06	3.28	5.02e-07 4.23
18	8.69e-05	1.31	2.94e-06	2.56	5.38e-07	3.20	1.77e-07 4.15
22	6.72e-05	1.28	1.77e-06	2.53	2.85e-07	3.16	7.74e-08 4.12
28	4.99e-05	1.23	9.69e-07	2.50	1.34e-07	3.13	2.91e-08 4.05
40	3.23e-05	1.22	4.01e-07	2.47	4.45e-08	3.08	

TABLE 5.1

The errors  $|||U_h - u|||_{10}$  for different time mesh gradings with  $\alpha = 0.2$ . We observe convergence of order  $k^{(\alpha+1)\gamma} (= k^{1.2\gamma})$  for  $1 \leq \gamma \leq (m+1)/(\alpha+1)$  for  $m = 1, 2, 3$ .

graphically optimal convergence rates of the DPG-FE scheme (2.4) in the time direction on the non-uniformly graded meshes in (4.9). To do so, we take a relatively large number of subintervals in space and choose  $r$  (the degree of the approximate finite element solution in the spatial variable) appropriately so that the temporal errors are dominating. Thus, by Theorem 4.3, convergence of order  $O(k^{2m+\alpha})$  for  $\gamma \geq \frac{2m+1+\alpha}{1-\alpha}$  is anticipated. However, and as in **Example 1**, the numerical results in Table 5.3 illustrate more optimistic convergence rates. We observed a uniform global error

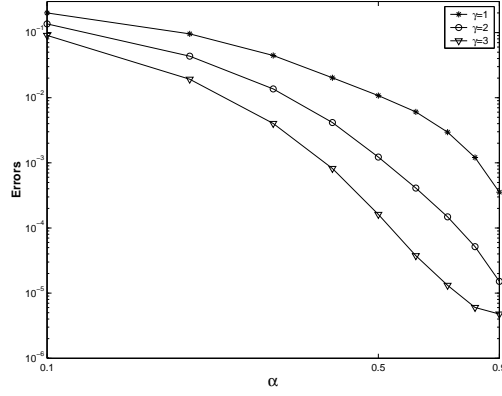


FIG. 5.2. The errors  $||U_h - u||_{10}$  for Example 1 plotted against  $\alpha$  for different values of  $\gamma$  and fixed  $N = 60$  with  $m = 2$ .

$N_x$	$r = 1$		$r = 2$		$r = 3$	
10	5.638e-03		1.576e-04		2.633e-06	
20	1.426e-03	1.983	1.796e-05	3.133	1.568e-07	4.069
30	6.367e-04	1.988	4.730e-06	3.291	2.998e-08	4.081
40	3.584e-04	1.998	1.979e-06	3.029	9.172e-09	4.117
60	1.592e-04	2.002	6.240e-07	2.847		

TABLE 5.2

The errors  $||U_h - u||_{10}$  for Example 1 with  $\alpha = 0.5$ . We observe a spatial convergence of order  $h^{r+1}$  for  $r = 1, 2, 3$ . The time mesh will be chosen such that the spatial errors are dominating.

bounded by  $Ck^{\min\{\gamma(1-\alpha), m+1\}}$  for  $\gamma \geq 1$ , which is optimal for  $\gamma \geq (m+1)/(1-\alpha)$  (relaxed) and not for  $\gamma \geq \frac{2m+\alpha+1}{1-\alpha}$  as the theory suggested. For graphical illustrations, see Figure 5.3.

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$N$	$\gamma = 1$		$\gamma = 2$		$\gamma = 3$		$\gamma = 3.6$		$\gamma = 4.2$	
	$m = 1$									
20	9.3e-03		1.5e-03		1.5e-03					
40	5.6e-03	0.73	4.5e-04	1.7	3.9e-04	2.0				
80	3.5e-03	0.67	1.7e-04	1.4	9.9e-05	2.0				
160	2.2e-03	0.68	6.5e-05	1.4	2.5e-05	2.0				
320	1.4e-03	0.68	2.5e-05	1.4	5.7e-06	2.1				
	$m = 2$									
10	6.3e-03		1.4e-03		3.2e-04		2.4e-04		3.7e-04	
20	3.8e-03	0.71	5.2e-04	1.5	6.7e-05	2.3	3.3e-05	2.9	3.1e-05	3.6
30	2.9e-03	0.69	2.9e-04	1.4	2.8e-05	2.2	1.1e-05	2.6	7.5e-06	3.5
40	2.4e-03	0.69	1.9e-04	1.4	1.5e-05	2.2	5.5e-06	2.6	2.9e-06	3.3
50	2.0e-03	0.69	1.4e-04	1.4	9.2e-06	2.1	3.1e-06	2.6	1.5e-06	3.1
80	1.5e-03	0.69	7.2e-05	1.4	3.4e-06	2.1	9.2e-07	2.6		

TABLE 5.3

The errors  $|||U_h - u|||_{10}$  for different time mesh gradings with  $m = 1$  and  $m = 2$  (that is, piecewise linear and piecewise quadratic DPG time stepping solution), and  $\alpha = 0.3$ . We observe convergence of order  $k^{(1-\alpha)\gamma} (= k^{0.7\gamma})$  for  $1 \leq \gamma \leq (m+1)/(1-\alpha)$ .

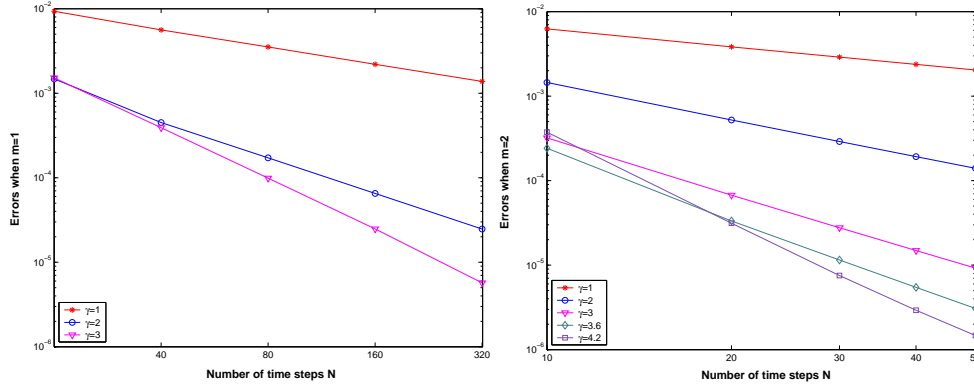


FIG. 5.3. The errors  $|||U_h - u|||_{10}$  for Example 2 plotted against  $N$  for different choices of  $\gamma$  and for  $m = 2, 3$ , with  $\alpha = 0.3$ .

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